Monotonic transformations and preferences

Given the utility functions $u(q_1, q_2) = (q_1 + 1)^{\frac{1}{3}} q_2$, $v(q_1, q_2) = 3 \ln(q_1 + 1) + \ln q_2$, and $w(q_1, q_2) = \ln(q_1 + 1)^{\frac{1}{6}} + \ln q_2^{\frac{1}{8}}$, show that those functions represent the same preferences.

Solution

It is known that the utility function, as a numerical representation of the order of preferences, is unique except for its monotonic transformations. Given the utility function $u(\cdot)$, the function $v(\cdot)$ is nothing more than $v(q_1, q_2) = \ln u$, with which it represents the same preference ordering as $u(\cdot)$ and thus gives rise to the same goods demands as $u(\cdot)$. In turn, the function $w(\cdot)$ can be written as

$$w = (q_1+1)^{\frac{1}{3}} q_2^{\frac{1}{3}} = \left[(q_1+1)^{\frac{1}{3}} q_2^{\frac{1}{3}}\right] = \exp\left\{\ln\left[(q_1+1)^{\frac{1}{3}} q_2^{\frac{1}{3}}\right]\right\} = \exp\left\{\frac{1}{3}\left[3\ln(q_1+1)+\ln q_2\right]\right\} = \exp\left(\frac{1}{3}v\right).$$

$$z = (x/y)^3 - 2^y x^2 y + e^2$$

Since additionally $\frac{dw}{dv} = \frac{1}{3} \cdot e^{\frac{1}{3}v} > 0$, given that the exponential function is a growing function, w is a growing transformation of v. Hence, the demand functions derived from the order of preferences represented by w are the same as those derived from the function v. On the other hand, given $v = \ln u$, $w = e^{\frac{1}{3}v} = e^{\frac{1}{3}\ln u} = u^{\frac{1}{3}}$, thus the function w is also a monotonic transformation of u. Therefore, the demands resulting from the preferences represented by w will coincide with the derivatives of the preferences represented by u.

Lastly, the function z can be expressed as

$$z = \frac{\ln(q_1+1)^6 + \ln q_2^2}{8} = \frac{\ln\left[(q_1+1)^6 \cdot q_2^2\right]}{8} = \frac{3\ln(q_1+1)+\ln q_2}{4}$$

And since $\frac{dz}{dv} = \frac{1}{4} > 0$, **it is also guaranteed that the demands arising from the function z will coincide with the demands from v, since the preference order represented by z is the same as that represented by v. Similarly, since $v = \ln u$, it follows that $z = \frac{1}{4}\ln u$, and the demands arising from the utility function z will be the same as those that derive from u.**